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Stability Properties and Asymptotics for *N* Nonminimally Coupled Scalar Fields Cosmology

L. Brenig,[1,5] A. Figueiredo,[2] E. Gunzig,[1,3] T. M. Rocha Filho,[2] and A. Saa[1,4]

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We consider here the dynamics of some homogeneous and isotropic cosmological models with N interacting classical scalar fields nonminimally coupled to the spacetime curvature, as an attempt to generalize some recent results obtained for one and two scalar fields. We show that a Lyapunov function can be constructed under certain conditions for a large class of models, suggesting that chaotic behavior is ruled out for them. Typical solutions tend generically to the empty de Sitter (or Minkowski) fixed points, and the previous asymptotic results obtained for the one field model remain valid. In particular, we confirm that, for large times and a vanishing cosmological constant, even in the presence of the extra scalar fields, the universe tends to an infinite diluted matter dominated era.

KEY WORDS: stability; asymptotics; scalar fields.

1. INTRODUCTION

We have considered recently the homogeneous and isotropic solutions of the cosmological model described by the action (Gunzig *et al.*, 2001b):

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left\{-(1-\xi\kappa\psi^2)\frac{R}{\kappa} + g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - 2V(\psi)\right\}, \qquad (1)$$

where ψ is the nonminimally coupled scalar field, $\kappa = 8\pi G$, G is the Newtonian constant, R the scalar curvature of spacetime, and the self-interaction potential $V(\psi)$ has the form

$$V(\psi) = \frac{3\alpha}{\kappa}\psi^2 - \frac{\Omega}{4}\psi^4 - \frac{9\omega}{\kappa^2}, \qquad (2)$$

[1] RggR, Université Libre de Bruxelles, CP 231, Brussels, Belgium.
[2] Instituto de Física, Universidade de Braília, DF, Brazil.
[3] Instituts Internationaux de Chimie et de Physique Solvay, CP 231, Brussels, Belgium.
[4] IMECC – UNICAMP, C.P. 6065, Campinas, SP, Brazil.
[5] To whom correspondence should be addressed at RggR, Université Libore de Bruxelles, CP 231, 1050 Brussels, Belgium; e-mails: lbrenig@ulb.ac.be.

with $\alpha = \frac{\kappa m^2}{6}$, m the mass of the scalar field, Ω is an arbitrary constant, and $-\frac{9\omega}{\kappa^2}$ is the usual cosmological constant Λ. The nonminimal coupling term in (1) is, of course, $\xi R\psi^2$ where ξ is an arbitrary constant. The results previously obtained (see Gunzig *et al.*, 2001a, for the motivations and references) were worked out for the case $\xi = 1/6$, the so-called conformally coupled case, and they point toward some novel and very interesting dynamical behavior: superinflation regimes, a possible avoidance of big-bang and big-crunch singularities through classical birth of the universe from empty Minkowski space, spontaneous entry into and exit from inflation, and a cosmological history suitable for describing quintessence in principle. Through exhaustive numerical simulations and with some semianalytical tools, the three-dimensional phase space of the model has been constructed. The existence of a Lyapunov function for the fixed points was crucial for the study of the asymptotic behavior (Gunzig *et al.*, 2001a) and for precluding the appearance of any chaotic regime, confirming, and shedding some light on some other previous results in this line (Gunzig *et al.*, 2000).

The robustness of its predictions must be an essential feature of any realistic cosmological model. The study of some generalizations of the model (1) is, therefore, mandatory. In (Figueiredo *et al.*, 2002), we consider the implications of the inclusion of a second interacting massless scalar field in the dynamics of homogeneous and isotropic solutions. The corresponding phase space becomes now five-dimensional, and richer structures could appear. In spite of this, we show that for a class of physically reasonable quartic interaction potentials, the neighborhoods of the relevant fixed points are unaltered, and, in particular, the asymptotic regimes obtained for the 1-field case (Gunzig *et al.*, 2001a) are preserved in the presence of an extra massless field. The relaxing of isotropy was the next performed robustness test of the model. In fact, it appeared as a corollary of a much more general result. We studied the singularities in the time-evolution of homogeneous and anisotropic solutions of cosmological models described by the action (Abramo *et al.*, 2003; Brenig *et al.*, manuscript submitted for publication):

$$S = \int d^4x\sqrt{-g}\{F(\psi)R - \partial_a\psi\partial^a\psi - 2V(\psi)\}, \tag{3}$$

with general $F(\psi)$ and $V(\psi)$. Such a model has been recently considered also in Esposito-Farese and Polarski (2001). We showed that anisotropic solutions generically evolve toward a space–time singularity that corresponds to the hypersurface $F(\psi) = 0$ in the phase space. Such singularity is harmless for isotropic solutions. A second, and different, type of singularity corresponds to the hypersurface $F_1(\psi) = 0$, where

$$F_1(\psi) = F(\psi) + \frac{3}{2}(F'(\psi))^2, \tag{4}$$

and no solution can avoid it. Starobinski (1981) was the first to identify the singularity corresponding to the hypersurfaces $F(\psi) = 0$, for the case of conformally coupled anisotropic solutions. Futamase and coworkers (Futamase *et al.*, 1981; Futamase and Maeda, 1989) identified both singularities in the context of chaotic inflation in $F(\psi) = 1 - \xi\psi^2$ theories (See also Bertolami, 1987; Deser, 1984; Hosotani, 1985). For this type of coupling, the first singularity is always present for $\xi > 0$ and the second one for $0 < \xi < 1/6$. The results of Abramo *et al.* (2003) and Brenig *et al.* (manuscript submitted for publication) are, however, more general since the case of general $F(\psi)$ is treated and all conclusions are based on the analysis of true geometrical invariants. The phase space for the model (3) is also five-dimensional, and many new structures appear. The appearance of the singularity of the first type implies the instability of the anisotropic solutions even for the conformally coupled case, for instance. The presence of any amount of anisotropy, no matter how small, makes the model unstable, challenging its validity as a realistic cosmological model. The asymptotic behavior in the neighborhood of the fixed points far from the hypersurface $F(\psi) = 0$ is, however, preserved.

We present here a new robustness test for the homogeneous and isotropic solutions for models of the type (1). We consider the case of several interacting scalar fields conformally coupled to space–time curvature. Exact solutions for this case are much more difficult to find, in particular we could not identify the heteroclinics and homoclinics as it was done (Gunzig *et al.*, 2001b). We could, however, get a strong result about the stability of fixed points suggesting that chaotic regimes are ruled out. This is proved for a class of interaction potentials, through the construction of an explicit Lyapunov function. In spite of the unstable character of anisotropies for models of the type (1), the results reported here are expected to be valid in the neighborhood of the Minkowski fixed point, even for the anisotropic case.

2. THE LYAPUNOV FUNCTION

Let us consider the action for N classical interacting scalar fields $\psi_1, \ldots, \psi_N$ conformally coupled to the space–time curvature,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left\{-\left[1 - \frac{\kappa}{6}\left(\psi_1^2 + \cdots + \psi_N^2\right)\right]\frac{R}{\kappa} + \sum_{i=1}^{N} g^{\mu\nu}\partial_\mu\psi_i\partial_\nu\psi_i - 2V(\psi_1, \ldots, \psi_N)\right\} \qquad (5)$$

where the potential V is a polynomial in the N fields variables ψ_i up to fourth degree. We consider homogeneous and isotropic solutions corresponding to

Robertson–Walker metrics with flat spatial section

$$ds^2 = d\tau^2 - a^2(\tau)(dx^2 + dy^2 + dz^2). \tag{6}$$

Variations of the action (1) with respect to the scalar fields ψ_i yield N coupled Klein–Gordon equations

$$\ddot{\psi}_i + 3H\dot{\psi}_i - \frac{1}{6}R\psi_i + \frac{\partial V}{\partial \psi_i} = 0, \tag{7}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble function and the dot denotes time derivative. The variation of S with respect to the metric leads to the Einstein equations which can be cast, in an analogous way to the one field case, in the form

$$\frac{\kappa}{2}\sum_{i=1}^{N}\dot{\psi}_i{}^2 + \kappa V - 3H^2 + \frac{\kappa}{2}H^2\sum_{i=1}^{N}\psi_i{}^2 + H\sum_{i=1}^{N}\psi_i\dot{\psi}_i = 0, \tag{8}$$

i.e. the energy constraint, and

$$R = -6(\dot{H} + 2H^2) = \kappa\left(-4V + \sum_{i=1}^{N}\psi_i\frac{\partial V}{\partial \psi_i}\right), \tag{9}$$

the trace equation. The system of ordinary differential Eqs. (7) and (9) is defined in a $(2N + 1)$-dimensional phase space. The solutions are, however, confined on the $2N$-dimensional zero energy hypersurface (8).

For physical interpretation reasons (the same ones as Gunzig *et al.*, 2001b), we assume that the potential $V(\psi_1, \ldots, \psi_N)$ contains a mass contribution in the form of a quadratic term for each field ψ_i, and a homogeneous quartic function $f_4(\psi_1, \ldots, \psi_N)$ describing the interactions of the N scalar fields (including possible self-interactions), and a cosmological constant

$$V(\psi_1, \ldots, \psi_N) = \frac{3}{\kappa}\sum_{i=1}^{N}\alpha_i\psi_i^2 + f_4(\psi_1, \ldots, \psi_N) - \frac{9\omega}{\kappa^2}, \tag{10}$$

where $\alpha_i = \frac{\kappa}{6}m_i^2$, m_i being the mass of field ψ_i.

We now proceed to show the existence of a Lyapunov function for the dynamical system (7)–(9) under some well-defined conditions on the potential (10). We focus on the subsystem of Klein-Gordon equations (7). By multiplying each equation by ψ_i and summing on the N fields we get

$$\frac{d}{d\tau}\left[\sum_{i=1}^{N}\frac{\dot{\psi}_i}{2} + V\right] - \frac{1}{6}R\sum_{i=1}^{N}\psi_i\dot{\psi}_i = -3H\sum_{i=1}^{N}\dot{\psi}_i{}^2. \tag{11}$$

Assuming that $-\frac{1}{6}R\psi_i$ derives from a potential

$$-\frac{1}{6}R\psi_i = \frac{\partial U}{\partial \psi_i}, \tag{12}$$

we would obtain

$$\frac{d}{d\tau}\left[\sum_{i=1}^{N}\frac{\dot{\psi_i}^2}{2}+V+U\right]=-3H\sum_{i=1}^{N}\dot{\psi_i}^2. \tag{13}$$

If the function between square brackets has a minimum at a given fixed point, it will be a candidate for a Lyapunov function for $H > 0$. Equations (7)–(9) have, obviously, many fixed points, but we are concerned only with de Sitter ($\omega < 0$) or Minkowski ($\omega = 0$) fixed points, for which $\psi_i = \dot{\psi} = 0$ and $H = \pm\sqrt{-3\omega/\kappa}$. We are restricted, therefore, to $\omega \leq 0$ (nonnegative cosmological constants, as in Gunzig *et al.*, 2001a,b). A closer analysis of the energy constraints (8) reveals that the system can cross the $2N$-hypersurface $H = 0$ only on a $2N - 1$ submanifold of the $2N + 1$ original phase space. In a neighborhood of the fixed point ($H = \sqrt{-3\omega/\kappa}$, $\psi_i = 0$) where V is nonnegative, the system can reach $H = 0$ only at the point where V vanishes, otherwise the system cannot reach $H = 0$. Hence, we will have for $H > 0$

$$\frac{d}{d\tau}\left[\frac{\dot{\psi_i}^2}{2}+V+U\right]\leq 0 \tag{14}$$

in such a neighborhood and, provided that $V + U$ has a minimum at the origin,

$$L=\sum_{i=1}^{N}\frac{\dot{\psi_i}^2}{2}+V+U \tag{15}$$

is a Lyapunov function, ensuring, thereby, the stability of the fixed point ($H = \sqrt{-3\omega/\kappa}$, $\psi_i = 0$).

The relevant question here is the validity of the hypothesis (12). It is valid, for instance, if all the fields have the same mass, $\alpha_1 = \alpha_2 = \ldots = \alpha_N$. This is a strong constraint that, as we will see later, can be somehow relaxed. From Eq. (9), with the potential given by (10), we have

$$-\frac{1}{6}R\psi_i=\psi_i\sum_{j=1}^{N}\alpha_j\psi_j^2-\frac{6\omega}{\kappa}\psi_i. \tag{16}$$

Condition (12) requires

$$\frac{\partial(R\psi_i)}{\partial\psi_k}=\frac{\partial(R\psi_k)}{\partial\psi_i}, \tag{17}$$

implying, from Eq. (9), that

$$\alpha_k\psi_i\psi_k=\alpha_i\psi_i\psi_k \tag{18}$$

for all couples (i, k) with $i \neq k$. The equal masses case is, obviously, a particular solution of (18). For this case, the potential U is readily shown to be

$$U(\psi_1, \psi_2, \ldots, \psi_N) = U_0 + \frac{\alpha}{4}|\psi|^4 - \frac{3\omega}{\kappa}|\psi|^2, \tag{19}$$

where U_0 is an arbitrary constant and $|\,|$ stands to the usual Euclidean norm

$$|\psi|^2 = \psi_i^2 + \ldots + \psi_N^2. \tag{20}$$

In this case, the Lyapunov function is given explicitly by

$$L = \frac{1}{2}|\dot{\psi}|^2 + \frac{3}{\kappa}(\alpha - \omega)|\psi|^2 + f_4(\psi_1, \ldots, \psi_N) - \frac{9\omega}{\kappa^2} + U_0 + \frac{\alpha|\psi|^4}{4} \tag{21}$$

for arbitrary homogeneous functions of degree four f_4. In summary, the function L has a minimum on the fixed point and there is a neighborhood (the attraction basin) with $H \geq 0$ where L has nonpositive time derivative. As the system can reach the hypersurface $H = 0$ only in the fixed point, this provides a stability proof for the fixed point $\dot{\psi}_i = \psi_i = 0$, $H = +\sqrt{3|\omega|/\kappa}$. An analysis of the time-reversed system reveals that the symmetric fixed point (with $H = -\sqrt{3|\omega|/\kappa}$) must be repulsive. The trajectories starting in the vicinity of the last point will leave this region and some of them will cross the hypersurface $H = 0$. Some of the crossing trajectories can eventually be trapped in the half-space $H > 0$ and will tend asymptotically toward the stable fixed point. This phase portrait suggest a regular behavior, and it is very reminiscent of the behavior exhibited by the solutions of the one-field model, where chaotic regimes were ruled out.

As it was already noted, this reasoning breaks down for N scalar fields with distinct masses. Nevertheless, we now prove that, under rather weak conditions relating the masses and the parameters for some potentials of the form (10), another Lyapunov function exists and ensures the stability of the fixed point. We will consider here the special case where the homogeneous function f_4 has the form (10)

$$f_4(\psi_1, \ldots, \psi_N) = -\sum_{l=1}^{N} \frac{\Omega_l}{4}\psi_l^4 + \sum_{k,l=1(k\neq l)}^{N} \left(\alpha_{lk}\psi_l\psi_k^3 + \frac{\beta_{lk}}{4}\psi_l^2\psi_k^2\right), \tag{22}$$

with $\beta_{lk} = \beta_{kl}$. The Klein–Gordon equations (7) become

$$\ddot{\psi}_i + 3H\dot{\psi}_i + \frac{6}{\kappa}(\alpha_i - \omega)\psi_i + (\alpha_i - \Omega_i)\psi_i^3 + F_i = 0, \tag{23}$$

where

$$F_i = \psi_i \sum_{j=1(j\neq i)}^{N} (\alpha_j + \beta_{ij})\psi_j^2 + \sum_{j=1(j\neq i)}^{N} \alpha_{ij}\psi_j^3 + 3\psi_i^3 \sum_{j=1(j\neq i)}^{N} \alpha_{ij}\psi_j. \tag{24}$$

Let us now rescale the field variables by a constant factor $\psi_i = \gamma_i \tilde{\psi}_i$. Klein–Gordon equations are now

$$\ddot{\tilde{\psi}}_i + 3H\dot{\tilde{\psi}}_i + \frac{6}{\kappa}(\alpha_i - \omega)\tilde{\psi}_i + \gamma_i^2(\alpha_i - \Omega_i)\tilde{\psi}_i{}^3 + \tilde{F}_i = 0, \tag{25}$$

with

$$\tilde{F}_i = \tilde{\psi}_i {\sum_{j=1}^{N}}' \gamma_j^2(\alpha_j + \beta_{ij})\tilde{\psi}_j{}^2 + \frac{1}{\gamma_i}{\sum_{j=1}^{N}}' \gamma_j \tilde{\psi}_j{}^3 + 3\gamma_i \tilde{\psi}_i{}^2 {\sum_{j=1}^{N}}' \gamma_j \alpha_{ij} \tilde{\psi}_j. \tag{26}$$

We again multiply both sides of (25) by $\tilde{\psi}_i$ and sum over i leading to

$$\frac{d}{d\tau}\left[\sum_{i=1}^{N} \frac{\dot{\tilde{\psi}}_i{}^2}{2} + V_1\right] + \sum_{i=1}^{N} \tilde{F}_i \dot{\tilde{\psi}} = -3H \sum_{i=1}^{N} \dot{\tilde{\psi}}_i{}^2, \tag{27}$$

where

$$V_1 = \sum_{i=1}^{N}\left[\frac{3}{\kappa}(\alpha_i - \omega)\tilde{\psi}_i{}^2 + \frac{\gamma_i^2}{4}(\alpha_i - \Omega_i)\tilde{\psi}_i{}^4\right]. \tag{28}$$

Let us determine the conditions under which the term $\sum_{i=1}^{N} \tilde{F}_i \dot{\tilde{\psi}}_i$ is a total derivative $\frac{dV_2}{d\tau}$. This requires that $\tilde{F}_i$ derives from a potential V_2, that is

$$\frac{\partial \tilde{F}_i}{\partial \tilde{\psi}_k} = \frac{\partial \tilde{F}_k}{\partial \tilde{\psi}_i}. \tag{29}$$

Conditions (29) lead to the following algebraic relations between the masses, the coefficients of the potential V and the scaling factor γ_i,

$$\gamma_i^2(\alpha_i + \beta_{ik}) = \gamma_k^2(\alpha_k + \beta_{ik}), \tag{30}$$

and $\alpha_{ik} = 0$. The conditions (30) provide sign conditions

$$\frac{\alpha_i + \beta_{ik}}{\alpha_k + \beta_{ik}} \geq 0 \tag{31}$$

and homogeneous algebraic linear equations for the γ_i^2 whose compatibility conditions relate directly the masses α_i with the interaction coupling constant β_{ik}. For $N = 2$ (two scalar fields), the conditions (30) are rather weak

$$\frac{\alpha_1 + \beta_{12}}{\alpha_2 + \beta_{12}} \geq 0. \tag{32}$$

For $N = 3$, these conditions provide the inequalities (31) and one strict equality

$$\frac{(\alpha_1 + \beta_{12})(\alpha_2 + \beta_{23})(\alpha_3 + \beta_{31})}{(\alpha_1 + \beta_{13})(\alpha_2 + \beta_{21})(\alpha_3 + \beta_{32})} = 1. \tag{33}$$

It is important to remind that $\beta_{ij} = \beta_{ji}$. For $N = 4$, there are four independent conditions

$$\frac{(\alpha_1 + \beta_{12})(\alpha_2 + \beta_{23})(\alpha_3 + \beta_{31})}{(\alpha_1 + \beta_{13})(\alpha_2 + \beta_{21})(\alpha_3 + \beta_{32})} = 1$$
$$\frac{(\alpha_2 + \beta_{23})(\alpha_3 + \beta_{34})(\alpha_4 + \beta_{42})}{(\alpha_2 + \beta_{24})(\alpha_3 + \beta_{32})(\alpha_4 + \beta_{43})} = 1$$
$$\frac{(\alpha_1 + \beta_{12})(\alpha_2 + \beta_{24})(\alpha_4 + \beta_{41})}{(\alpha_1 + \beta_{14})(\alpha_2 + \beta_{12})(\alpha_4 + \beta_{24})} = 1$$
$$\frac{(\alpha_1 + \beta_{14})(\alpha_3 + \beta_{31})(\alpha_4 + \beta_{43})}{(\alpha_1 + \beta_{13})(\alpha_3 + \beta_{34})(\alpha_4 + \beta_{41})} = 1. \tag{34}$$

There are four conditions and eight parameters α_i, β_{ij}. More generally, the total number of parameters α_i, β_{ij} is $\frac{N^2}{2}$ and the total number of conditions (strict equalities) is $\frac{N(N-1)(N-2)}{6}$. We, thus, can determine the maximum number of interacting scalar fields above which the above conditions are overdetermined

$$\frac{N^2}{2} < \frac{N(N-1)(N-2)}{6}, \tag{35}$$

which yields the solution $N > 5$. Finally, the explicit expression of the Lyapunov function in the original variables ψ_i is

$$L = \frac{1}{2}\sum_{i=1}^{N}\frac{1}{\gamma_i^2}\dot{\psi}_i{}^2 + \frac{3}{\kappa}\sum_{i=1}^{N}\left(\frac{\alpha_i - \omega}{\gamma_i^2}\right)\psi_i^2 + \frac{1}{4}\sum_{i=1}^{N}\left(\frac{\alpha_i - \omega_i}{\gamma_i^2}\right)\psi_i^4$$
$$+\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1(j\neq i)}^{N} N\left(\frac{\alpha_i - \beta_{ij}}{\gamma_j^2}\right)\psi_i^2\psi_k^2, \tag{36}$$

with the conditions (30) over the parameters α_i, β_{ij} and γ_i^2.

Again, this function vanishes at the fixed points and on the H-axis. It is non–negative definite in a finite domain of the phase space around that axis and its time derivative is given by

$$\frac{dL}{d\tau} = -3H\sum_{i=1}^{N}\frac{1}{\gamma_i^2}\dot{\psi}_i{}^2, \tag{37}$$

which is non-positive in the half-space $H \geq 0$. This proves the stability of the fixed point $\dot{\psi}_i = \psi_i = 0$. $H = \sqrt{\frac{3|\omega|}{\kappa}}$. All the solutions tend to this point or go to infinity. Furthermore, both Lyapunov functions may be global, i.e.,

$$\lim_{|\psi|\to\infty} L = +\infty, \tag{38}$$

when the parameters satisfy some inequalities. In that case, the Lyapunov stability theory shows that solutions are bounded and accumulate on one of the fixed points, depending on the initial conditions.

3. CONCLUSION

We have shown under which conditions some N nonminimally scalar fields homogeneous and isotropic cosmological models admit a Lyapunov function for their de Sitter (or Minkowski) fixed point. The physical interpretation of such conditions is still unclear.

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